Lecture 13

2020A
Fall, 2020

- line integral of first kind

Recall that a vector in $\mathbb{R}^3$ is just a point written as

$$\vec{x} = (x, y, z)$$
$$\overset{\text{or}}{=} (x_1, x_2, x_3).$$

Its norm or length is given by

$$|\vec{x}| = \sqrt{x^2 + y^2 + z^2}.$$

A vector is called a unit vector if $|\vec{x}| = 0$. For any non-zero vector, we have the polar decomposition

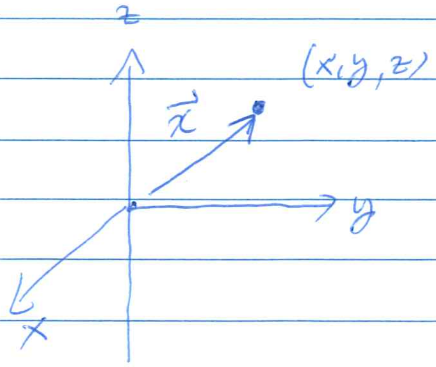
$$\vec{x} = |\vec{x}| \, \frac{x}{|\vec{x}|}.$$

Note that

$$\hat{i} = (1,0,0), \quad \hat{j} = (0,1,0), \quad \hat{k} = (0,0,1).$$

We have

$$\vec{x} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k},$$



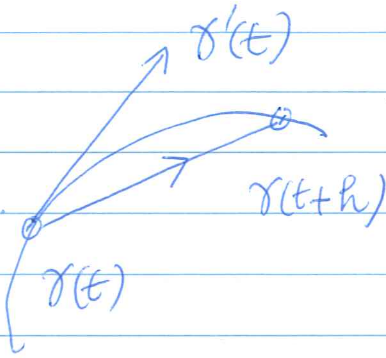
A parametric curve is a continuous map from some interval to $\mathbb{R}^3$. Usually we take the interval to be $[a, b]$.

$$\gamma : [a, b] \to \mathbb{R}^3, \quad \gamma(t) = (x(t), y(t), z(t))$$
$$\overset{\text{or}}{=} (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

γ_i conti. $\forall i$.

It is called C^1 if γ_i' exist and are continuous. One can see that $\vec{\gamma}'$ points to the tangent direction (when $\vec{\gamma}' \neq (0,0,0)$).

A parametric curve is called regular if it is C^1 and $|\vec{\gamma}'(t)| > 0 \quad \forall t \in [a,b]$.



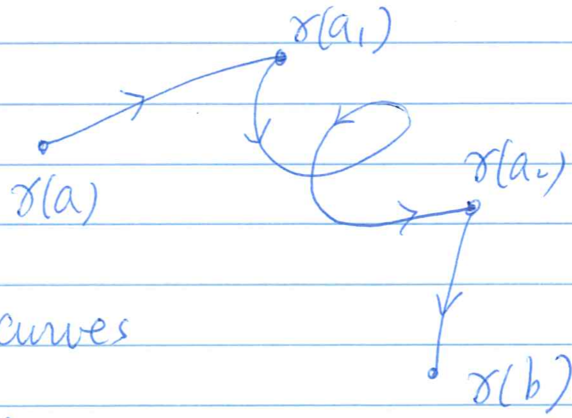
(Sometimes one allows $|\vec{\gamma}'| = 0$ at finitely many points in $[a,b]$.)

e.g. $\gamma : [0,1] \to \mathbb{R}^3$, $\gamma(t) = (a,b,c)$. then $\gamma'(t) = (0,0,0)$

so it is C^1 but not regular.

Roughly speaking, a regular curve is a C^1-curve such that, as t runs monotonically from a to b, its image also keeps running monotonically from $\gamma(a)$ to $\gamma(b)$.

A piecewise regular parametric curve $\gamma : [a,b] \to \mathbb{R}^3$ is such that $\exists\, a_1, \dots, a_n \in (a,b)$ s.t. $\gamma|_{[a_j, a_{j+1}]}$ is a regular curve.



Given $\gamma_1 : [a,b] \to \mathbb{R}^3$,
$\gamma_2 : [b,c] \to \mathbb{R}^3$, C^1-curves

under the condition $\gamma_1(b) = \gamma_2(b)$, the curve

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [a,b] \\ \gamma_2(t), & t \in [b,c], \end{cases}$$

defines a piecewise C^1-curve on $[a,c]$. Write $\gamma = \gamma_1 + \gamma_2$.

The image of a parametric curve $\gamma([a,b])$ is a subset in $\mathbb{R}^3$.

Let f be a continuous fcn defined on $\gamma([a,b])$. We'd like to define the line integral of f along γ. Using the idea of Riemann sums, let P be a partition on $[a,b]$:

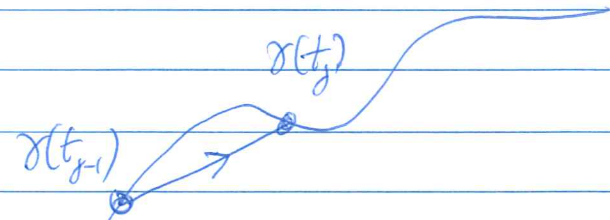
$$a = t_0 < t_1 < \cdots < t_n = b,$$

the Riemann sums are

$$\sum_j f(x(t_j^*), y(t_j^*), z(t_j^*))\,\Delta s_j$$

where Δs_j is the distance between $\gamma(t_{j-1})$ and $\gamma(t_j)$. We use it to approximate the arc length between $\gamma(t_{j-1})$ & $\gamma(t_j)$.

$$\Delta s_j = |\vec{\gamma}(t_j) - \vec{\gamma}(t_{j-1})|$$



$$= \sqrt{(x(t_j)-x(t_{j-1}))^2 + (y(t_j)-y(t_{j-1}))^2 + (z(t_j)-z(t_{j-1}))^2}$$

$$= \sqrt{x'(\tilde{t}_j)^2 + y'(\hat{t}_j)^2 + z'(t_j^{\star})^2}\;\Delta t_j, \quad \tilde{t}_j, \hat{t}_j, t_j^{\star} \in [t_{j-1}, t_j]$$

$\therefore$ Riemann sums become

$$\sum_j f(x(t_j^*), y(t_j^*), z(t_j^*)) \sqrt{x'(\tilde{t}_j)^2 + y'(\hat{t}_j)^2 + z'(t_j^{\star})^2}\;\Delta t_j.$$

all tag pts $t_j^*, \tilde{t}_j, \hat{t}_j, t_j^{\star}$ can be replaced by, say, t_j with ~~negligible error, hence as $\|P\| \to 0$, the Riemann sums become~~

$$\int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt.$$

We define the line integral of f along the parametric curve γ to be

$$\int_\gamma f\,ds = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt.$$

Here it is assumed γ to be C^1. For a piecewise C^1-curve, we break it up $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ where each γ_j is C^1 and set

$$\int_\gamma f\,ds = \int_{\gamma_1} f\,ds + \cdots + \int_{\gamma_n} f\,ds.$$

All parametric curves considered in this chapter are piecewise C^1 and 1-1 onto its image.

When γ is regular, from the consideration above we see that

- $\int_\gamma f\,ds$ is the total mass of the image of γ when $f \geq 0$ is its density,

- $\int_\gamma ds$ is the length of γ, or more precisely, the length of the image of γ.

Consider the following parametric curves:

$\gamma_1 : [0, 2\pi] \to \mathbb{R}^2, \quad \gamma_1(t) = (\cos t, \sin t),$

$\gamma_2 : [0, 2\pi] \to \mathbb{R}^2, \quad \gamma_2(t) = (\sin t, -\cos t),$

$\gamma_3 : [0, 2\pi] \to \mathbb{R}^2, \quad \gamma_3(t) = (\cos 2t, \sin 2t),$

$\gamma_4 : [0, 3\pi] \to \mathbb{R}^2, \quad \gamma_4(t) = (\cos t, \sin t),$

$\gamma_5 : [-1, 1] \to \mathbb{R}^2, \quad \gamma_5(x) = (x, \sqrt{1-x^2}),$

$\gamma_6 : [1, 3] \to \mathbb{R}^2, \quad \gamma_6(x) = (2-x, -\sqrt{1-(2-x)^2}).$

You can check that the image of $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 + \gamma_6$ are the same — the unit circle. All of them are regular. But only the first two are 1-1 onto (except at the endpoints).

$|\gamma_j'(t)| = 1, \; j = 1, 2, 4, \; |\gamma_3'(t)| = 2$

$$\int_{\gamma_1} ds = \int_{\gamma_2} ds = 2\pi, \quad \int_{\gamma_3} ds = 4\pi, \quad \int_{\gamma_4} ds = 3\pi.$$

γ_2 differs from γ_1 by its orientation, γ_1 runs anticlockwise but γ_2 clockwise.

A curve is a geometric object in people's mind. Now a parametric curve is a map! How to make us more comfortable about the situation? we need to define a (geometric) curve. Well, a geometric curves should be a subset in $\mathbb{R}^2$ or $\mathbb{R}^3$ looks very thin. It has no volume, no area but has length. In practice, a curve in $\mathbb{R}^2$ is defined in terms of equations, For instance, the set

$$\{(x,y) : x^2+y^2 = r^2\}$$

① straight lines in $\mathbb{R}^2$:

$$ax+by=c, \quad \gamma(t)=(x(t),y(t))$$

$$x(t)=t,\ y(t)=\frac{1}{b}(c-at)$$

$$t\in(-\infty,\infty).$$

when $b=0$, $\gamma(t)=\left(\frac{a}{c},t\right)$

② straight lines in $\mathbb{R}^3$: e.g. $x-y+2=0$, $x+3y=1$,

$$\gamma(t)=\left(\frac{1}{4}(1-\frac{3}{4}t),\ \frac{1}{4}(1+t),\ t\right), \quad t\in(-\infty,\infty)$$

③ circle in $\mathbb{R}^2$: $(x-a)^2+(y-b)^2=r^2$,

$$\gamma(t)=(a+r\cos t,\ b+r\sin t),$$

$$t\in[0,2\pi].$$

④ parabola in $\mathbb{R}^2$: $y=ax^2+b$,

$$\gamma(t)=(t,\ at^2+b), \quad t\in(-\infty,\infty).$$

⑤ In general, the graph of a fcn $f=f(x)$ is the set

$$\{(x,f(x)) : x\in[a,b]\}$$

is a curve, and the standard parametrization is

$$\gamma(t)=(t,f(t)), \quad t\in[a,b].$$

It is C^1 iff f is C^1, & $|\gamma'(t)|=\sqrt{1+f'(t)^2}$.

the length of this graph is

$$\int_\gamma ds = \int_a^b \sqrt{1+f'^2(t)}\, dt .$$

A natural question arises: Let γ_1 and γ_2 be two regular parametric curves whose images are the same. Moreover, they map 1-1 onto its target. Do they have the same integral?

Theorem 1 Let $\gamma : [a,b] \to \mathbb{R}^3$, $\eta : [c,d] \to \mathbb{R}^3$ be two regular parametric curves 1-1 onto the same image. then

$$\int_\gamma ds = \int_\eta ds.$$

Pf: Need to show

$$\int_a^b |\gamma'(t)|\, dt = \int_c^d |\eta'(z)|\, dz .$$

Let P be a partition on $[a,b]$: $a < t_1 < t_2 < \cdots < t_n = b$. $\gamma(a), \gamma(t_1), \cdots, \gamma(t_n)$ become pts on the image curve. Since the correspondence is 1-1 onto, we can find $z_0 = c < z_1 < \cdots < z_n = d$ such that $\gamma(a) = \eta(c)$, $\gamma(t_j) = \eta(z_j)$, $\gamma(b) = \eta(d)$. Moreover, $\|P\| \to 0$ implies $\|Q\| \to 0$ where $Q : z_0 = c < z_j \cdots < d$.

$$\int_a^b |\gamma'(t)|\, dt \sim \sum_j \Big[(\gamma_1(t_j) - \gamma_1(t_{j-1}))^2 + (\gamma_2(t_j) - \gamma_2(t_{j-1}))^2$$

defines a circle of radius r. The set

$$\{(x,y) : y = ax^2+1,\ a>0\}$$

defines a parabola. In $\mathbb{R}^3$, a curve is the intersection of two surfaces. For instance

$$\{(x,y,z) : x^2+y^2+z^2=1,\ x+y+z=0\}$$

is a circle on the sphere $x^2+y^2+z^2=1$. The set

$$\{(x,y,z) : x^2+y^2+z^2=4,\ z=x^2+6y^2-1\}$$

also defines a curve.

Summing up, plane curves are usually described by

$$\{(x,y) : f(x,y)=0\}$$

and space curves by

$$\{(x,y,z) : f(x,y,z)=0,\ g(x,y,z)=0\}.$$

The implicit function theorem we learnt in 2010 provides a criterion on f and (f,g) when these sets really define a geometric object called curves.

Whenever a (geometric) curve is given, we know how to parametrize it, that is, to write down a parametric curve so that its image is the given curve. Moreover, the correspondence is 1-1 (except some pts.)

Here are some examples.

$$+ (\gamma_3(t_j) - \gamma_3(t_{j-1}))^2 \Big]^{1/2}$$

$$= \sum_j \Big[(\eta_1(z_j) - \eta_1(z_{j-1}))^2 + (\eta_2(z_j) - \eta_2(z_{j-1}))^2 + (\eta_3(z_j) - \eta_3(z_{j-1}))^2 \Big]^{\frac{1}{2}}$$

$$\sim \int_c^d |\eta'(z)|\, dz. \quad \#$$

Thm 1 tells us that in order to find the length of a curve, we can introduce a parametrization of it, that is, a parametric curve so that it maps 1-1 onto its image which is the given curve. Then perform the line integral to get the arc length of the given curve. The evaluation is independent of the choice of parametrization.

From thm 1 we deduce the following result. First of all, let $\gamma : [a, b] \to \mathbb{R}^3$

be a parametric curve. A reparametrization of γ is

$$\tilde{\gamma} : [\alpha, \beta] \to \mathbb{R}^3$$

where $\tilde{\gamma} = \gamma \circ \varphi$, $\varphi : [\alpha, \beta] \to [a, b]$ 1-1 onto C^1-map with C^1-inverse.

<u>Theorem 2</u> $\int_\gamma ds$ <u>is the same under reparametrization</u>, ie

$$\int_\gamma ds = \int_{\tilde{\gamma}} ds.$$

We also have the following interesting fact.

Theorem 3 There is a reparametrization for a regular parametric curve such that $|\eta'(s)| = 1$.

Pf. Let $\gamma : [a, b] \to \mathbb{R}^3$ be a regular parametric curve.

Let
$$s = \int_a^t |\gamma'(z)|\, dz .$$

Since $|\gamma'(z)| > 0$, as t runs monotonically from a to b, s runs monotonically from 0 to L (the arc length of $\gamma([a,b])$)

then φ be the inverse of $t \mapsto s$. Now, $\varphi : [0, L] \to [a, b]$

and we set $\eta(s) = \gamma \circ \varphi(s)$.

$$\eta'(s) = \gamma'(t) \frac{d\varphi}{ds} = \gamma'(t) \frac{1}{\frac{ds}{dt}} = \frac{\gamma'(t)}{|\gamma'(t)|}$$

$\therefore |\eta'(s)| = 1$, done #